B. MATH (HONS.) IST YEAR
FINAL EXAM
ELEMENTARY NUMBER THEORY
9TH DECEMBER, 2022
TOTAL MARKS - 50

Instruction to students. Please present your solutions as clearly as possible. Any theorem/result that you use directly should be clearly stated/mentioned.

(1) (i) If p is an odd prime, then show that $(p-1)! \equiv \left(1.2\ldots(\frac{p-1}{2})\right)^2(-1)^{(p-1)/2}(\text{mod } p)$. Hence deduce that the congruence equation $x^2 \equiv -1(\text{mod } p)$ (where p is any prime) has a solution if and only if $p = 2$ or $p \equiv 1(\text{mod } 4)$. *Hint* : Use Wilson's theorem and Fermat's little theorem. $(2+3)$

(ii) Let $n \geqslant 2$ be an integer and $k = 2^n + 1$. Show that if $3^{(k-1)/2} + 1 \equiv 0(\text{mod } k)$, then k is a prime. (3)

(iii) Prove that the only positive integers dividing both n^2+1 and $(n+1)^2+1$ for some integer n are 1 and 5. *Hint* : Show that any such positive integer must necessarily divide both $4n+7$ and $2n+1$. (3)

(2) (i) Prove that the Legendre symbol $\left(\frac{561}{659}\right) = 1$. (4)

(ii) Prove that if n is an odd positive integer, then every prime divisor of $2^n - 1$ is of the form $8k \pm 1$. (3)

(iii) Show that if $p = 2^n + 1$ is a prime for some $n \geqslant 2$, then $3^{(p-1)/2} + 1$ is divisible by p. *Hint* : Use Euler's criterion and law of quadratic reciprocity. (4)

(iv) Show that a prime $p > 3$ satisfies the congruence $x^2 \equiv -3(\text{mod } p)$ if and only if $p \equiv 1(\text{mod } 6)$. *Hint* : Use law of quadratic reciprocity. (4)

(3) (i) If m, n are positive integers such that $\gcd(m, n) = 1$, then show that the set of positive divisors of mn consists of all products d_1d_2, where d_1 divides m, d_2 divides n and gcd $(d_1, d_2) = 1$. (2)

(ii) Define a multiplicative function. If F is multiplicative and $F(n) = \sum_{d|n} f(d)$, then show that f is also multiplicative. $(1+3)$

(iii) Let $\tau(n) = \sum_{d|n} 1$, the number of divisors of n. Show that τ is a multiplicative function and hence deduce that $\tau(n)$ is odd if and only if n is a perfect square. $(1+2)$

(4) (i) Find all positive integral solutions of $9x + 7y = 200$. (3)

(ii) Define $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}\ \forall\ n \geqslant 2$. Determine the characteristic polynomial of this linear recurrence relation and hence show that $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\ \forall\ n \in \mathbb{N}$. $(1+3)$

(5) (i) For $a, b, c, d \in \mathbb{N}$, show that $(a^2 + b^2)(c^2 + d^2)$ can be written as a sum of two squares. Hence express 54145 as a sum of two squares. $(1+3)$

(ii) Show that a positive integer n can be expressed as the difference of two squares if and only if n is not of the form $4k+2$. *Hint* : If n is not of the form $4k+2$, then consider the two cases $n \equiv 1$ or $3 (\text{mod } 4)$ and $n \equiv 0 (\text{mod } 4)$ separately. (4)